"Comment on Canonical Transformation Applications to Optimal Trajectory Analysis"

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In their recent contribution[1] to the analysis of optimal trajectories, Professors Powers and Tapley open interesting fields of applications for the classical techniques of celestial mechanics. By the present Note we mean to support their contention. In particular we would like to show how the traditional Delaunay's variables help in giving explicitly—without quadratures—the complete solution to the basic Hamilton-Jacobi equation for coasting arcs.

We address ourselves to the Hamiltonian function

$$K = \lambda_r \cdot R + \lambda_\theta \cdot \Theta/r^2 + \lambda_R \cdot (\Theta^2 - \mu r)/r^3 \qquad (1)$$

defined in an eight-dimensional phase space, product of a four-dimensional space of coordinates (r,θ,R,Θ) by a four-dimensional space of momenta $(\lambda_r,\lambda_\theta,\lambda_R,\lambda_\Theta)$. It may be helpful to remember that, in the problem studied by Professors Powers and Tapley, the variables (r,θ) are polar coordinates while R is the radial component of the velocity, and Θ the angular momentum per unit of mass.

We propose to transfer from the polar phase variables (r,θ,R,Θ) to Delaunay's phase variables (l,g,L,G). Our task is then to extend Delaunay's mapping into a completely canonical transformation from the phase space $(r,\theta,R,\Theta,\lambda_r,\lambda_\theta,\lambda_R,\lambda_\Theta)$ into a phase space of the four-dimensional coordinate space (l,g,L,G) by a four-dimensional momentum space $(\lambda_l,\lambda_g,\lambda_L,\lambda_G)$. But we know[2] that a homogeneous extension will do. As a matter of fact from the differential identity

$$\lambda_r \cdot dr + \lambda_\theta \cdot d\theta + \lambda_R \cdot dR + \lambda_\Theta \cdot d\Theta = \lambda_l \cdot dl + \lambda_g \cdot dg + \lambda_L \cdot dL + \lambda_G \cdot dG$$

defining a homogeneous extension, we deduce that

$$\lambda_r = \frac{\partial l}{\partial r}\cdot\lambda_l + \frac{\partial g}{\partial r}\cdot\lambda_g + \frac{\partial L}{\partial r}\cdot\lambda_L + \frac{\partial G}{\partial r}\cdot\lambda_G$$

$$\lambda_\theta = \frac{\partial l}{\partial \theta}\cdot\lambda_l + \frac{\partial g}{\partial \theta}\cdot\lambda_g + \frac{\partial L}{\partial \theta}\cdot\lambda_L + \frac{\partial G}{\partial \theta}\cdot\lambda_G$$

$$\lambda_R = \frac{\partial l}{\partial R}\cdot\lambda_l + \frac{\partial g}{\partial R}\cdot\lambda_g + \frac{\partial L}{\partial R}\cdot\lambda_L + \frac{\partial G}{\partial R}\cdot\lambda_G$$

$$\lambda_\Theta = \frac{\partial l}{\partial \Theta}\cdot\lambda_l + \frac{\partial g}{\partial \Theta}\cdot\lambda_g + \frac{\partial L}{\partial \Theta}\cdot\lambda_L + \frac{\partial G}{\partial \Theta}\cdot\lambda_G$$

But, Delaunay's transformation being canonical, the symplectic character of its Jacobian of partial derivatives implies[3] that

$$\frac{\partial l}{\partial r} = \frac{\partial R}{\partial L}, \quad \frac{\partial l}{\partial \theta} = \frac{\partial \Theta}{\partial L}, \quad \frac{\partial l}{\partial R} = -\frac{\partial r}{\partial L}, \quad \frac{\partial l}{\partial \Theta} = -\frac{\partial \theta}{\partial L}$$

$$\frac{\partial g}{\partial r} = \frac{\partial R}{\partial G}, \quad \frac{\partial g}{\partial \theta} = \frac{\partial \Theta}{\partial G}, \quad \frac{\partial g}{\partial R} = -\frac{\partial r}{\partial G}, \quad \frac{\partial g}{\partial \Theta} = -\frac{\partial \theta}{\partial G}$$

$$\frac{\partial L}{\partial r} = -\frac{\partial R}{\partial l}, \quad \frac{\partial L}{\partial \theta} = -\frac{\partial \Theta}{\partial l}, \quad \frac{\partial L}{\partial R} = \frac{\partial r}{\partial l}, \quad \frac{\partial L}{\partial \Theta} = \frac{\partial \theta}{\partial l}$$

$$\frac{\partial G}{\partial r} = -\frac{\partial R}{\partial g}, \quad \frac{\partial G}{\partial \theta} = -\frac{\partial \Theta}{\partial g}, \quad \frac{\partial G}{\partial R} = \frac{\partial r}{\partial g}, \quad \frac{\partial G}{\partial \Theta} = \frac{\partial \theta}{\partial g}$$

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Consequently the equations of the extension become

$$\lambda_r = \frac{\partial R}{\partial L}\cdot\lambda_l + \frac{\partial R}{\partial G}\cdot\lambda_g - \frac{\partial R}{\partial l}\cdot\lambda_L - \frac{\partial R}{\partial g}\cdot\lambda_G$$

$$\lambda_\theta = \frac{\partial \Theta}{\partial L}\cdot\lambda_l + \frac{\partial \Theta}{\partial G}\cdot\lambda_g - \frac{\partial \Theta}{\partial l}\cdot\lambda_L - \frac{\partial \Theta}{\partial g}\cdot\lambda_G$$

$$\lambda_R = -\frac{\partial r}{\partial L}\cdot\lambda_l - \frac{\partial r}{\partial G}\cdot\lambda_g + \frac{\partial r}{\partial l}\cdot\lambda_L + \frac{\partial r}{\partial g}\cdot\lambda_G$$

$$\lambda_\Theta = -\frac{\partial \theta}{\partial L}\cdot\lambda_l - \frac{\partial \theta}{\partial G}\cdot\lambda_g + \frac{\partial \theta}{\partial l}\cdot\lambda_L + \frac{\partial \theta}{\partial g}\cdot\lambda_G$$

There remains now to produce the partial derivatives of Delaunay's transformation. As such a list is not yet to be found in the standard references, we shall reproduce here the expressions in the form we found to be most convenient. Note that we designate by f the true anomaly;

$$\frac{\partial r}{\partial l} = \frac{L^3}{G^3}\, re\left(\sin f + \frac{1}{2}\, e \sin 2f\right)$$

$$\partial r/\partial g = 0$$

$$\frac{\partial r}{\partial L} = \frac{r}{Le}\left(\frac{3}{2}\, e - \cos f - \frac{1}{2}\, e \cos 2f\right)$$

$$\frac{\partial r}{\partial G} = \frac{r}{Ge}\left(\frac{1}{2}\, e + \cos f + \frac{1}{2}\, e \cos 2f\right)$$

$$\frac{\partial \theta}{\partial l} = \frac{L^3}{G^3}\left[\left(1 + \frac{1}{2}\, e^2\right) + 2e \cos f + \frac{1}{2}\, e^2 \cos 2f\right]$$

$$\partial\theta/\partial g = 1$$

$$\frac{\partial \theta}{\partial L} = \frac{1}{Le}\left(2 \sin f + \frac{1}{2}\, e \sin 2f\right)$$

$$\frac{\partial \theta}{\partial G} = -\frac{1}{Ge}\left(2 \sin f + \frac{1}{2}\, e \sin 2f\right)$$

$$\frac{\partial R}{\partial l} = \frac{\mu L^3 e}{G^4}\,\left[e + \left(1 + \frac{3}{4}e^2\right)\cos f + e \cos 2f + \frac{1}{4}e^2 \cos 3f\right]$$

$$(\partial R/\partial g) = 0$$

$$\frac{\partial R}{\partial L} = \frac{\mu}{GLe}\left[\left(1 - \frac{3}{4}\, e^2\right)\sin f + e \sin 2f + \frac{1}{4}\, e^2 \sin 3f\right]$$

$$\frac{\partial R}{\partial G} = -\frac{\mu}{G^2 e}\left[\left(1 + \frac{1}{4}\, e^2\right)\sin f + e \sin 2f + \frac{1}{4}\, e^2 \sin 3f\right]$$

$$(\partial\Theta/\partial l) = (\partial\Theta/\partial g) = (\partial\Theta/\partial L) = 0$$

$$(\partial\Theta/\partial G) = 1$$

Elementary but somewhat lengthy manipulations will then convert the Hamiltonian (1) into the surprisingly simple function

$$K = (\mu^2/L^3)\cdot\lambda_l \qquad (2)$$

Consequently the coast-arc problem is described by the differential equations

$$\dot{l} = (\partial K/\partial\lambda_l) = (\mu^2/L^3),\ \dot{\lambda}_l = -(\partial K/\partial l) = 0$$

$$\dot{g} = (\partial K/\partial\lambda_g) = 0,\ \dot{\lambda}_g = -(\partial K/\partial g) = 0$$

$$\dot{L} = (\partial K/\partial\lambda_L) = 0,\ \dot{\lambda}_L = -(\partial K/\partial L) = 3(\mu^2/L^4)\lambda_l$$

$$\dot{G} = (\partial K/\partial\lambda_G) = 0,\ \dot{\lambda}_G = -(\partial K/\partial G) = 0$$

In other words, we recover the constants of an elliptic Keplerian motion, namely Delaunay's action L, the angular momentum G and the argument of perigee g; we retrieve the mean anomaly l as the usual linear form $nt + l_0$, where l_0 is the mean anomaly at epoch. Moreover, we find that the multipliers λ_l, λ_g, λ_G are constants.

We have not attempted to recover from here the solution given by Professors Powers and Tapley, for there is enough information given in their note and in this comment to carry out this exercise to a successful conclusion.

Of course Delaunay's transformation is restricted to elliptic Keplerian motions. There are known universal Delaunay's variables, but we have not succeeded yet in producing for them a set of convenient partial derivatives. Hence we do not know yet how to produce the solution of the coast-arc in universal variables along the lines we have taken here for elliptic motions.

References

[1] Powers, W. F. and Tapley, B. D., "Canonical Transformation Applications to Optimal Trajectory Analysis," *AIAA Journal*, Vol. 7, No. 3, March 1969, pp. 394–399.

[2] Whittaker, E. T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, Cambridge, England, 1960, pp. 301–302.

[3] Tisserand, F., *Traité de Mécanique Céleste*, Tome 1, Gauthier-Villars, Paris, 1889, pp. 20–23.

Reply by Authors to A. Deprit

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The solution to the elliptic coast-arc problem that Dr. Deprit has developed is indeed an interesting way to attack the problem. The purpose of this Reply is to point out some of the similarities and differences of the two solutions.

First, the solution of Ref. 1 was developed in terms of Poincare variables because, as is well known, they are well defined at circular conditions, whereas the Delaunay variables involve the argument of perigee that is undefined at circular conditions. Since many of the space guidance problems of interest involve circular or near-circular conditions at one point or another, it was imperative that we obtain a solution that is valid for such cases.

Second, Eqs. (35) and (36) of Ref. 1 are actually transformation equations between the Poincare variables and the set $\{\alpha_1, \ldots, \alpha_5, \beta_1, \ldots, \beta_5\}$, which represents a full set of constant parameters for the coast-arc. One can use Eqs. (23) and (A1) to obtain the transformation equations between the polar variables and the coast-arc parameters $\{\alpha,\beta\}$. The resultant set of equations would then be analogous to the list of transformations in Dr. Deprit's paper. In the $\{\alpha,\beta\}$ system, the variational Hamiltonian on the coast-arc is $K \equiv 0$, whereas $K = (\mu^2/L^3)\lambda_l$ on the coast-arc in Dr. Deprit's solution. The essential fact is that even though both resultant Hamiltonians have simple forms, the transformation equations between the polar system and either orbital parameter system still are cumbersome. However, if one need not transform back to the polar system, then both systems have desirable properties.

Third, the solution of Ref. 1 may be extended easily to take into account nonplanar angles and multipliers, e.g., i and Ω. For example, if $q_6 \equiv i$, $q_7 \equiv \Omega$, $p_6 \equiv \Lambda_i$, $p_7 \equiv \Lambda_\Omega$, then $(dq_6/d\theta) = 0$ and $(dq_7/d\theta) = 0$ on the coast-arc and the Hamiltonian for the coast-arc [i.e., Eq. (29)] is unchanged. Thus, it may be shown[2] that if the S-function of Eq. (34) is denoted by S^*, then $S = S^* + \alpha_6 q_6 + \alpha_7 q_7$ is a complete solution of the new Hamilton-Jacobi equation for the coast arc.

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References

[1] Powers, W. F. and Tapley, B. D., "Canonical Transformation Applications to Optimal Trajectory Analysis," *AIAA Journal*, Vol. 7, No. 3, March 1969, pp. 394–399.

[2] Powers, W. F., "Hamiltonian Perturbation Theory for Optimal Trajectory Analysis," Rept. EMRL TR-1003, June 1966, The University of Texas Engineering Mechanics Research Lab., Austin, Texas.

Comments on "Study of Nonlinear Systems"

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In recent Technical Notes B. V. Dasarathy and P. Srinivasan have outlined methods of solving certain types of nonlinear differential systems of third- and second-order by use of transformation functions. In their paper, it is shown that use of nonlinear transformation function actually reduces the nonlinear system into an equivalent linear system, which can easily be solved to obtain the response of the original system. The transformations indicated in their Refs. 1 and 2 is the transformation of both independent and dependent variables. This transformation technique is applied to solve the nonlinear system of the type given below

$$Md^2x/dt^2 + Cx^n(dx/dt) + K_1x^n + K_2x^{2n+1} = 0$$

where M, C, K_1 and K_2 are constant parameters.

When this is applied the previous equation is transformed into a linear system and thus gives an implicit relation between independent and dependent variables in terms of a common parameter.

A study of the nonlinear equation chosen by the aforementioned authors, revealed a certain relationship between the coefficient of dx/dt and the other functions of x in the equation. As a result of this property, the equation can be integrated. This Note presents two approaches to solve the nonlinear system, by use of this property.

Ist Approach

The equation to be solved is

$$M(d^2x/dt^2) + Cx^n(dx/dt) + K_1x^n + K_2x^{2n+1} = 0 \quad (1)$$

This can be written as

$$(d^2x/dt^2) + f(x)(dx/dt) + F(x) = 0 \quad (2)$$

where

$$f(x) = (C/M)x^n$$

$$F(x) = (1/M)(K_1x^n + K_2x^{2n+1})$$

By change of dependent variable to independent variable, we get

$$(d^2t/dx^2) = f(x)(dt/dx)^2 + F(x)(dt/dx)^3$$

Since the transformed equation has dependent variable as t and this does not appear in the equation as such the following substitution can be made $R = dt/dx$, and we have

$$dR/dx = f(x)R^2 + F(x)R^3 \quad (3)$$

Equation (3) is of "Ablesche" type. By close observation of the coefficient of R^2 and R^3, it is seen that they satisfy the following relation, viz.,

$$(d/dx)[F(x)/f(x)] = \lambda f(x)$$

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